Int J Theor Phys (2011) 50: 95–105
DOI 10.1007/s10773-010-0497-9

Sufficient and Necessary Conditions of Entanglement Transformations Between Mixed States

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Received: 29 June 2010 / Accepted: 17 September 2010 / Published online: 1 October 2010

Abstract Based on a new measure of entanglement for finite-dimensional bi-particle pure states, we give sufficient and necessary conditions that a bi-particle mixed state ρ can be transformed into another mixed state σ by local operations and classical communication (LOCC). This result can be regarded as a generalization of Nielsen's theorem (Nielsen, Phys. Rev. Lett. 83:436, 1999). However, we find that it is more difficult to determine the entanglement transformations between mixed states than to do between pure ones.

Keywords Transformation · Mixed state · Entanglement

1 Introduction

In recent years majorization has received renewed attention in quantum information theory, mainly as a result of Nielsen's discovery that exposes the necessary and sufficient conditions for pure bi-particle entanglement transformations [1]. According to Nielsen's theorem, $|\phi\rangle \xrightarrow{LOCC} |\psi\rangle \Leftrightarrow \lambda_\phi \prec \lambda_\psi$, that is, the transformation between two pure bi-particle states $|\phi\rangle \to |\psi\rangle$ can be performed by means of local operations and classical communication (LOCC) if and only if $\lambda_\phi \prec \lambda_\psi$, where λ_ϕ denotes the vector of Schmidt coefficients of $\mathrm{tr}_A(|\phi\rangle\langle\phi|)$. This result has been extended by Jonathan and Plenio [2] to the case where the probabilistic transformation $|\phi\rangle \xrightarrow{LOCC} \{p_i, |\psi_i\rangle\}$ can be accomplished iff $\lambda_\phi \prec \sum_i p_i \lambda_{\psi_i}$. Moreover, according to the Schmidt decomposition [3], the reduced density operators ρ_A and ρ_B have the same Schmidt number. We hence consider that the entanglement of pure bi-particle states are completely determined by their Schmidt coefficients. So the Von Neumann entropy has been accepted as a canonical measure of entanglement for pure bi-particle states.

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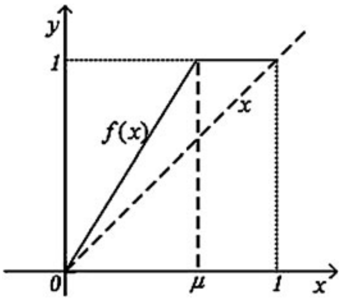

Fig. 1 The graphs of the function $f(x)$ and x

However, in practical application people have to deal with mixed states rather than pure ones due to decoherence. This begs the question: Is majorization a suitable tool for transformation from one mixed state into another one? If it is not true, what is the condition of $\rho \xrightarrow{LOCC} \sigma$? A function $f : \mathbf{R}^d \to \mathbf{R}$ is said to be Schur-convex if $x \prec y \Longrightarrow f(x) \leq f(y)$ [4]. The consequence of this result is that the Von Neumann entropy is a Schur-concave, that is, $\lambda_\rho \prec \lambda_\sigma \Longrightarrow S(\rho) \geq S(\sigma)$. However, the Schumucher noiseless coding theorem for quantum information [5] implies that the maximum rate of error-free quantum information transmission is $S(\rho)$ qubits per signal. If $S(\rho) \geq S(\sigma)$ holds, then clearly the coding theorem will be violated [6]. Now we have to find other measurements to quantify the entanglement of mixed states. Gour considered an essentially different type of measure of entanglement for two-qubit quantum systems [7]. For any two-qubit pure state $|\varphi\rangle = \sqrt{\lambda_0}|00\rangle + \sqrt{\lambda_1}|11\rangle$ the author defined the entanglement as follows:

$$E_\mu(|\varphi\rangle) = f_\mu(x) = \begin{cases} \frac{x}{\mu} & \text{for } x \leq \mu, \\ 1 & \text{for } x > \mu \end{cases}$$

with $\forall\ 0 < \mu \leq 1$, where $x = 2\min\{\lambda_0, \lambda_1\}$. In fact x is the infimum of $f(x)$, that is, $x = \inf_{\mu\in[0,1]} f(x)$ (see Fig. 1).

Based on this definition, Gour gave necessary and sufficient conditions for entanglement transformation between two probability distributions of two-qubit pure states by LOCC. However, this result cannot be viewed as the entanglement transformation between two-qubit mixed states by LOCC. For example, let $\rho = \frac{1}{2}|\Phi^+\rangle\langle\Phi^+| + \frac{1}{2}|\Phi^-\rangle\langle\Phi^-|$ and $\sigma = |\Phi^+\rangle\langle\Phi^+|$ with $|\Phi^\pm\rangle = \frac{|00\rangle\pm|11\rangle}{\sqrt{2}}$. It is well known that ρ cannot be transformed to σ by LOCC, although they still meet the condition mentioned in Ref. [7]. In fact, the main reason is that $f_\mu(x)$ is no longer competent to measure the entanglement of $d \times d$-dimensional pure states. In this paper, we first define a new entanglement measure for finite-dimensional bi-particle pure states, then present the necessary and sufficient conditions for entanglement transformation of $d \times d$-dimensional mixed states, where d is arbitrary finite number.

2 The Condition for Entanglement Transformation Between $d \times d$-Dimensional Mixed States

We first summarize briefly some basic concepts and results that are needed for further treatment. An ensemble of pure states $\mathcal{E}$, which is usually represented by $\{p_i, |\psi_i\rangle\}$, is characterized by a finite set of positive numbers p_i ($\sum_i p_i = 1$) and a corresponding set of normalized vectors $|\psi_i\rangle$ of the Hilbert space $\mathcal{H}$ [8]. The density operator ρ (a trace, semi-definite positive operator) associated to $\{p_i, |\psi_i\rangle\}$ is defined as:

$$\rho = \sum_{i=1} p_i |\psi_i\rangle\langle\psi_i|, \qquad \sum_{i=1} p_i = 1, \quad p_i \geq 0. \tag{1}$$

As is well known, the correspondence between ensembles and density operators is definitely many to one. In the light of this point, let us consider the set of all ensembles of quantum systems as a whole. Consequently, such a set can be naturally endowed with an equivalence relation. Let $[\rho]$ denote the set of ensembles corresponding the density operator ρ.

Definition 2.1 [9] We say that two sets of ensembles $[\rho]$ and $[\rho']$ are equivalent iff $\rho = \rho'$.

Remark 2.2 It is obvious that the relation defined above is reflexive, symmetric and transitive, and it leads to a partition of all ensembles.

Following the ideas presented in [7], we first discuss the entanglement transformation between two ensembles $\{p_i, |\psi_i\rangle\} \xrightarrow{LOCC} \{q_j, |\phi_j\rangle\}$. Suppose Alice and Bob share a pure state $|\psi\rangle$, and then perform a quantum operation ε which outputs the pure states $|\psi_i\rangle (i = 1, 2, \ldots, n)$ with probability p_i. We then consider that Alice and Bob obtain an ensemble $\{p_i, |\psi_i\rangle\}$. Alice and Bob perform again quantum operations ε_i' which output the pure states $|\phi_j\rangle$ with conditional probability p_{ji} for all the possible outcome states, that is, $\varepsilon_i'(|\psi_i\rangle\langle\psi_i|) = \sum_j p_{ji}|\phi_j\rangle\langle\phi_j|$. Thus, the transformation $\varepsilon' = (\varepsilon_1', \varepsilon_2', \ldots, \varepsilon_n')$, defined between the ensembles $\{p_i, |\psi_i\rangle\}$ and $\{q_j, |\phi_j\rangle\}$, outputs the states $|\phi_j\rangle$ with probability $q_j = \sum_i p_i p_{ji}$, that is, $\varepsilon'(\sum_i p_i|\psi_i\rangle\langle\psi_i|) = \sum_{i,j} p_i p_{ji}|\phi_j\rangle\langle\phi_j|$ with $q_j = \sum_i p_i p_{ji}$, where $\sum_j p_{ji} = 1$.

In the rest of this paper, we discuss the mixed states from the state space $\mathcal{B}(\mathcal{H}^{\otimes 2})$ over d-dimensional Hilbert space $\mathcal{H}$.

Definition 2.3 Let $x = (x_1, x_2, \ldots, x_d)$ be the vector of decreasingly ordered Schmidt coefficients of $\mathrm{tr}_A(|\psi\rangle\langle\psi|)$ in $\mathcal{B}(\mathcal{H})$. We define $\hat{E}_j(|\psi\rangle)$ as follows:

$$\hat{E}_j(|\psi\rangle) = \sum_{i=j}^{d} \frac{x_i}{k} \wedge 1, \quad j = 2, 3, \ldots, d, \ \forall k \in (0, 1]. \tag{2}$$

Remark 2.4 It is easily found that $\hat{E}_j(|\psi\rangle)$ is the infimum of the entanglement monotone which was defined in Ref. [10]. Indeed, it is an entanglement monotone. In fact, for any $j = 2, 3, \ldots, d$ we have $\sum_i p_i \hat{E}_j(|\psi_i\rangle) = \sum_{i,j} \frac{p_i x_{ij}}{k} \wedge p_i \geq \hat{E}_j(\sum_i p_i|\psi_i\rangle)$.

For an arbitrary mixed state $\rho \in \mathcal{B}(\mathcal{H}^{\otimes 2})$, we take

$$\hat{E}_j(\rho) = \min_{\substack{\{p_l, |\psi_l\rangle\} \\ \rho = \sum_l p_l|\psi_l\rangle\langle\psi_l|}} \sum_l p_l \hat{E}_j(|\psi_l\rangle) \quad (j = 2, 3, \ldots d). \tag{3}$$

For brevity, we shall write $\min_t \sum_l p_l^{(t)} \hat{E}_j(|\psi_l^{(t)}\rangle)$ instead of the right hand of (3). Since $\sum_{i=1}^{d} x_{li}^{(t)} = 1$ for fixed t and l, we can choose $d - 1$ elements of $x_{li}^{(t)}$ as free variables.

As $\hat{E}_j(|\psi\rangle)$ is an entanglement monotone, there exists $|\psi_l^{(t_d)}\rangle$ such that $\sum_l p_l(\frac{x_{ld}^{(t_{0d})}}{k} \wedge 1) = \min_t \sum_l p_l^{(t)}(\frac{x_{ld}^{(t)}}{k} \wedge 1)$ [11]. Similarly, we can find $|\psi_l^{(t_j)}\rangle (j = 2, 3, \ldots, d-1)$ such that $\sum_l p_l(\sum_{i=j}^d \frac{x_{li}^{(t_{0i})}}{k} \wedge 1) = \min_t \sum_l p_l^{(t)}(\sum_{i=j}^d \frac{x_{li}^{(t)}}{k} \wedge 1)$. Construct a set of $|\psi_l^{(t_0)}\rangle$ such that $(x_{l1}^{(t_{01})}, x_{l2}^{(t_{02})}, \ldots, x_{ld}^{(t_{0d})})$ are their vector of decreasingly ordered Schmidt coefficients, where $x_{li}^{(t_{0i})} (i = 1, 2, \ldots, d)$ meet the conditions that mentioned above. Therefore, we can make sure that there always exists at least an ensemble $\{p_i^{(t_0)}, |\psi_i^{(t_0)}\rangle\}$ such that $\hat{E}_j(\rho) = \sum_l p_l \hat{E}_j(|\psi_l\rangle)$.

Remark 2.5 An ensemble that achieves the minimum in (3) is referred to an optimal one in this paper.

Let $\{p_i, |\psi_i\rangle\}$ and $\{q_j, |\phi_j\rangle\}$ be two optimal ensembles of bi-particle mixed states ρ and σ in $\mathcal{B}(\mathcal{H}^{\otimes 2})$ over d-dimensional Hilbert space $\mathcal{H}$, respectively. We say

Definition 2.6 $\rho \xrightarrow{LOCC} \sigma$ if $\{p_i, |\psi_i\rangle\} \xrightarrow{LOCC} \{q_j, |\phi_j\rangle\}$.

Remark 2.7 We consider Definition 2.6 is reasonable. This means that $\rho \xrightarrow{LOCC} \sigma$ if each ensemble of ρ can be transformed into an optimal ensemble of σ by LOCC. Otherwise $\rho \nrightarrow \sigma$ by LOCC.

Returning to the problem of entanglement transformations between two bi-particle mixed states. We have

Theorem 2.8

$$\rho \xrightarrow{LOCC} \sigma \textit{ if and only if } \hat{E}_j(\rho) \geq \hat{E}_j(\sigma), \quad j = 2, 3, \ldots, d. \tag{4}$$

Proof It is easy to testify that the right hand of (4) satisfies the conditions of entanglement monotones. Since LOCC cannot increase the entanglement of state, $\hat{E}_j(\rho) \geq \hat{E}_j(\sigma)$ $(j = 2, 3, \ldots, d)$ holds. □

Having proved the necessity of the conditions for entanglement transformations, we shall demonstrate the sufficiency. We first consider the 4-dimensional case, which demonstrates the essential idea of the general proof. Assume that $\{p_i, |\psi_i\rangle\}$ $(i = 1, 2, \ldots, n)$ and $\{q_l, |\phi_l\rangle\}$ $(l = 1, 2, \ldots, m)$ are two optimal ensembles of arbitrary mixed states ρ and σ, respectively. Let $x_i = (x_{i1}, x_{i2}, x_{i3}, x_{i4})$ and $y_l = (y_{l1}, y_{l2}, y_{l3}, y_{l4})$ be the vector of decreasingly ordered Schmidt coefficients of $\mathrm{tr}_A(|\psi_i\rangle\langle\psi_i|)$ and $\mathrm{tr}_A(|\phi_l\rangle\langle\phi_l|)$, respectively. Then $\hat{E}_j(\rho) \geq \hat{E}_j(\sigma)$ $(j = 2, 3, 4)$ can be rewritten as

$$\sum_{i=1}^n p_i \left(\frac{x_{i4}}{k} \wedge 1\right) \geq \sum_{j=1}^m q_j \left(\frac{y_{j4}}{k} \wedge 1\right), \tag{5}$$

$$\sum_{i=1}^n p_i \left(\frac{x_{i3} + x_{i4}}{k} \wedge 1\right) \geq \sum_{j=1}^m q_j \left(\frac{y_{j3} + y_{j4}}{k} \wedge 1\right), \tag{6}$$

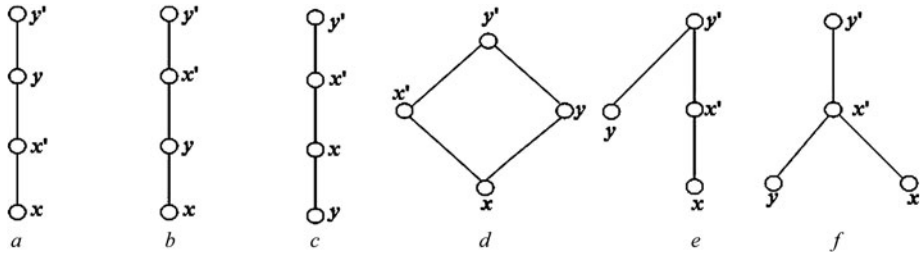

Fig. 2 The possible partial order relations of x, x', y, y'

$$\sum_{i=1}^{n} p_i\left(\frac{x_{i2}+x_{i3}+x_{i4}}{k}\wedge 1\right) \geq \sum_{j=1}^{m} q_j\left(\frac{y_{j2}+y_{j3}+y_{j4}}{k}\wedge 1\right) \tag{7}$$

with $\forall k \in (0,1]$. Without loss of generality, we assume $m=n=2$. For convenience, let $x=(x_1,x_2,x_3,x_4)$, $x'=(x'_1,x'_2,x'_3,x'_4)$, $y=(y_1,y_2,y_3,y_4)$ and $y'=(y'_1,y'_2,y'_3,y'_4)$ denote the vector of Schmidt coefficients of $\mathrm{tr}_A(|\psi_1\rangle\langle\psi_1|)$, $\mathrm{tr}_A(|\psi_2\rangle\langle\psi_2|)$, $\mathrm{tr}_A(|\phi_1\rangle\langle\phi_1|)$ and $\mathrm{tr}_A(|\phi_2\rangle\langle\phi_2|)$, which arranged into decreasing order, respectively. $\{p_i,|\psi_i\rangle\} \xrightarrow{LOCC} \{q_j,|\phi_j\rangle\}$ if and only if there exists a set of p_{li} such that $x \prec q_1 y + q_2 y'$ and $x' \prec q_1 y + q_2 y'$. According to [2], it means that there exists a set of p_{li} such that

$$x_4 \geq p_{11} y_4 + p_{21} y'_4, \tag{8}$$

$$x'_4 \geq p_{12} y_4 + p_{22} y'_4, \tag{8'}$$

$$x_3 + x_4 \geq p_{11}(y_3+y_4) + p_{21}(y'_3+y'_4), \tag{9}$$

$$x'_3 + x'_4 \geq p_{12}(y_3+y_4) + p_{22}(y'_3+y'_4), \tag{9'}$$

$$x_2 + x_3 + x_4 \geq p_{11}(y_2+y_3+y_4) + p_{21}(y'_2+y'_3+y'_4), \tag{10}$$

$$x'_2 + x'_3 + x'_4 \geq p_{12}(y_2+y_3+y_4) + p_{22}(y'_2+y'_3+y'_4). \tag{10'}$$

Note that $\{p_i,|\psi_i\rangle\} \xrightarrow{LOCC} \{q_l,|\phi_l\rangle\}$ becomes transformation between pure state and ensemble in the case of $x=x'$ [2]. We thus assume $x \neq x'$ and $y \neq y'$.

(I) $x \prec x'$, $y \prec y'$. This case implies that $x' \succ y'$. The reason is that at least one of the three inequalities $x'_4 \geq y'_4$, $x'_3+x'_4 \geq y'_3+y'_4$, $x'_2+x'_3+x'_4 \geq y'_2+y'_3+y'_4$ does not hold if $x' \nprec y'$. Without loss of generality, we assume $x'_4 < y'_4$. However this contradicts the fact that take $k=y'_4$ in (5). Since the majorization $\prec$ is a partial order relation [12], we now discuss the possible options according to the partial order of x, x', y, y' as follows:

(i) If $x' \prec y$. Since the majorization is transitive, we get $x \prec x' \prec y \prec y'$ (see Fig. 2(a)).

It is easy to prove the following fact:

Lemma 2.9 *If $x \prec y$ and $x \prec y'$, then $x \prec ty + (1-t)y'$ for any $t \in [0,1]$.*

By Lemma 2.9, $x \prec p_{11} y + p_{21} y'$ and $x' \prec p_{21} y + p_{22} y'$ hold for arbitrary p_{11} and p_{21}. This means $\{p_i,|\psi_i\rangle\} \xrightarrow{LOCC} \{q_j,|\phi_j\rangle\}$.

(ii) If $x \prec y \prec x' \prec y'$ (see Fig. 2(b)). According to Lemma 2.9, $x \prec ty + (1-t)y'$ for any $t \in [0,1]$. Thus, it is sufficient to testify that (8′)–(10′) hold. Note that (8) $\Leftrightarrow$

$$p_{12} \leq (x'_4 - y'_4)/(y_4 - y'_4). \tag{11}$$

Since $p_1 p_{11} + p_2 p_{12} = q_1$, (11) $\Leftrightarrow$

$$p_1 + p_2(x_4' - y_4')/(y_4 - y_4') \geq q_1. \tag{12}$$

The reason is that the following lemma holds:

Lemma 2.10 $\forall p_i, a_i$ $(i = 1, 2, \ldots, n) \in [0, 1]$, *if there exists* $a \in [0, 1]$ *such that* $\sum_{i=1}^n p_i a_i = a$, *then* $\sum_{i=1}^n p_i a_i' \geq a$ *if and only if there exists at least a set of* a_i *such that* $a_i \leq a_i'$ *for all* $1 \leq i \leq n$.

Proof Obviously $\sum_{i=1}^n p_i a_i' \geq a$ holds if we assume $a_i \leq a_i'$ for all $1 \leq i \leq n$. □

We induct on n. In this case $n = 2$, since $p_1 a_1' + p_2 a_2' \geq a$, we have $p_1(a_1' - a_1) + p_2(a_2' - a_2) \geq 0$. This implies that $a_1 \leq a_1'$ or $a_2 \leq a_2'$ holds. Without loss of the generality, we suppose $a_1 \leq a_1'$. Note that $p_2(a_2' - a_2) \geq p_1(a_1 - a_1')$. Take $a_1 = a_1'$, then $p_2(a_2' - a_2) \geq \max_{a_1}\{p_1(a_1 - a_1')\} = 0$. Furthermore, we can get a_2 such that $a_2 \leq a_2'$. Next we assume the result is true up to n, and try to prove it for $n + 1$. Similarly we have $\sum_{i=1}^n p_i(a_i' - a_i) \geq 0$. This implies that there exists at least an a_i such that $a_i \leq a_i'$ holds. Without loss of the generality, we suppose $a_1 \leq a_1'$. Note that $\sum_{i=2}^n p_i(a_i' - a_i) \geq p_1(a_1 - a_1')$. Take $a_i = a_i'$ $(i = 2, 3, \ldots, n)$, then $\sum_{i=2}^n p_i(a_i' - a_i) \geq \max_{a_1}\{p_1(a_1 - a_1')\} = 0$. The inductive hypothesis implies that there exist a_i such that $a_i \leq a_i'$ for all $1 \leq i \leq n$.

Equation (12) can be rewritten as:

$$p_1 y_4 + p_2 x_4' \geq q_1 y_4 + q_2 y_4'. \tag{13}$$

The equation can be got by taking $k = y_4$ in (5).

Similarly (9′) $\Leftrightarrow$

$$p_1 + p_2 \frac{(x_3' + x_4') - (y_3' + y_4')}{(y_3 + y_4) - (y_3' + y_4')} \geq q_1. \tag{14}$$

Equation (14) can be rewritten as:

$$p_1(y_3 + y_4) + p_2(x_3' + x_4') \geq q_1(y_3 + y_4) + q_2(y_3' + y_4'). \tag{15}$$

The equation can be got by taking $k = y_3 + y_4$ in (6). Equation (10′) $\Leftrightarrow$

$$p_1(y_2 + y_3 + y_4) + p_2(x_2' + x_3' + x_4') \geq q_1(y_2 + y_3 + y_4) + q_2(y_2' + y_3' + y_4'). \tag{16}$$

The equation can be got by taking $k = y_2 + y_3 + y_4$ in (7). Obviously (8′)–(10′) hold if set

$$p_{12} \leq \min\left\{\frac{x_4' - y_4'}{y_4 - y_4'}, \frac{(x_3' + x_4') - (y_3' + y_4')}{(y_3 + y_4) - (y_3' + y_4')}, \frac{(x_2' + x_3' + x_4') - (y_2' + y_3' + y_4')}{(y_2 + y_3 + y_4) - (y_2' + y_3' + y_4')}\right\}.$$

Hence (8′)–(10′) can be obtained by taking $k = y_4$, $y_3 + y_4$ and $y_2 + y_3 + y_4$ in (5)–(7).

(iii) If $y \prec x \prec x' \prec y'$ (see Fig. 2(c)). Note that (8) $\Leftrightarrow$

$$p_{11} \leq (x_4 - y_4')/(y_4 - y_4'). \tag{17}$$

According to Lemma 2.10, (8) and (8′) $\Leftrightarrow$

$$p_1 \frac{x_4 - y_4'}{y_4 - y_4'} + p_2 \frac{x_4' - y_4'}{y_4 - y_4'} \geq q_1. \tag{18}$$

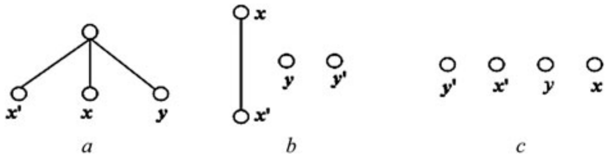

Fig. 3 The possible partial order relations of x, x', y, y'

Equation (18) can be rewritten as:

$$p_1 x_4 + p_2 x_4' \geq q_1 y_4 + q_2 y_4'. \tag{19}$$

The equation can be got by taking $k = 1$ in (5).

Similarly (9) and (9′) ⇔

$$p_1(x_3 + x_4) + p_2(x_3' + x_4') \geq q_1(y_3 + y_4) + q_2(y_3' + y_4'). \tag{20}$$

The equation can be got by taking $k = 1$ in (6). Equations (10) and (10′) ⇔

$$p_1(x_2 + x_3 + x_4) + p_2(x_2' + x_3' + x_4') \geq q_1(y_2 + y_3 + y_4) + q_2(y_2' + y_3' + y_4'). \tag{21}$$

The equation can be got by taking $k = 1$ in (7). Obviously (8)–(10) hold if set

$$p_{11} \leq \min\left\{\frac{x_4 - y_4'}{y_4 - y_4'}, \frac{(x_3 + x_4) - (y_3' + y_4')}{(y_3 + y_4) - (y_3' + y_4')}, \frac{(x_2 + x_3 + x_4) - (y_2' + y_3' + y_4')}{(y_2 + y_3 + y_4) - (y_2' + y_3' + y_4')}\right\}.$$

Hence (8)–(10) and (8′)–(10′) can be obtained by taking $k = 1$ in (5)–(7).

(iv) If $x' \parallel y$, that is, $x' \not\succ y$ and $y \not\succ x'$ (see Fig. 2(d)). This case implies that one or two of the following three inequalities $x_4' \geq y_4$, $x_3' + x_4' \geq y_3 + y_4$, $x_2' + x_3' + x_4' \geq y_2 + y_3 + y_4$ should not hold. At first, let us consider this case when one of the three inequalities does not hold. Without loss of the generality, we assume $x_3' + x_4' < y_3 + y_4$. Thus, we only consider the condition which (9) holds. Note that (9) ⇔

$$p_{12} \leq \frac{x_3' + x_4' - (y_3 - y_4)}{y_3 + y_4 - (y_3' + y_4')} \Leftrightarrow p_1 + p_2 \frac{x_3' + x_4' - (y_3 - y_4)}{y_3 + y_4 - (y_3' + y_4')} \geq q_1. \tag{22}$$

Equation(22) can be rewritten as (20). It can be got by taking $k = y_3 + y_4$ in (6).

When two of the three inequalities do not hold, without loss of the generality, we assume $x_4' < y_4$ and $x_2' + x_3' + x_4' < y_2 + y_3 + y_4$. Thus we consider the condition in which (8) and (10) hold. Similarly (8) ⇔ (13) can be got by taking $k = y_4$ in (5).

Similarly (10) ⇔ (21) can be got by taking $k = y_2 + y_3 + y_4$ in (7).

(v) If $x \prec x' \prec y'$, $y \prec y'$ and $x \parallel y$, $x' \parallel y$ (see Fig. 2(e)). This case is similar to (iv).

(vi) If $x \prec x'$, $y \prec y'$, $x \prec y'$ and $x \parallel y$ (see Fig. 2(f)). This case is similar to (iv).

(II) If $x \parallel x'$, $y \prec y'$. Although their partial order relations have seven possible options, it is sufficient to only consider the case of $x \parallel x'$, $x' \parallel y$ and $x \parallel y$ (see Fig. 3(a)).

It means that one or two of three of inequalities should not hold in the following set of inequalities, respectively:

$$\begin{cases} x_4' \geq x_4, \\ x_3' + x_4' \geq x_3 + x_4, \\ x_2' + x_3' + x_4' \geq x_2 + x_3 + x_4, \end{cases} \qquad \begin{cases} x_4 \geq y_4, \\ x_3 + x_4 \geq y_3 + y_4, \\ x_2 + x_3 + x_4 \geq y_2 + y_3 + y_4, \end{cases}$$

$$\begin{cases} x_4' \geq y_4, \\ x_3' + x_4' \geq y_3 + y_4, \\ x_2' + x_3' + x_4' \geq y_2 + y_3 + y_4. \end{cases}$$

For convenience, let the sign 1 denote that the inequality holds while 0 denote the inequality does not. For example, 101 denotes $x_4 \geq x'_4$, $x_3 + x_4 < x'_3 + x'_4$ and $x_2 + x_3 + x_4 \geq x'_2 + x'_3 + x'_4$. $(101)^{\mathrm{T}}$ denotes $x_4 \geq x'_4$, $x'_4 < y_4$ and $x_4 \geq y_4$. Furthermore, $(101)^{\mathrm{T}}(101)^{\mathrm{T}}(101)^{\mathrm{T}}$ denotes

$$\begin{cases} x'_4 \geq x_4, \\ x'_3 + x'_4 \geq x_3 + x_4, \\ x'_2 + x'_3 + x'_4 \geq x_2 + x_3 + x_4, \end{cases} \qquad \begin{cases} x_4 < y_4, \\ x_3 + x_4 < y_3 + y_4, \\ x_2 + x_3 + x_4 < y_2 + y_3 + y_4, \end{cases}$$

$$\begin{cases} x'_4 \geq y_4, \\ x'_3 + x'_4 \geq y_3 + y_4, \\ x'_2 + x'_3 + x'_4 \geq y_2 + y_3 + y_4. \end{cases}$$

It is not trivial to consider the possible options of these inequalities hold. In other words, we are to discuss the possible options of 0 and 1.

(i) Each column has only one 0 in $(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}$. There have 15 possible options of 0 and 1.
(ii) There exists one and only one column that has two 0 in $(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}$. There are $12 + 14 + 5 = 31$ possible options of 0 and 1.
(iii) There exist two columns that have two 0 in $(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}$. There are $6+15+15 = 36$ possible options of 0 and 1.
(iv) Each column has two 0 in $(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}$. There are 11 possible options of 0 and 1.

Without loss of the generality, we discuss the case $(011)^{\mathrm{T}}(110)^{\mathrm{T}}(011)^{\mathrm{T}}$, the other cases can be deduced similarly. In this case, we have $x'_4 > x_4 \geq y_4$, $x_3 + x_4 \geq x'_3 + x'_4 \geq y_3 + y_4$ and $x_2 + x_3 + x_4 \geq y_2 + y_3 + y_4 > x'_2 + x'_3 + x'_4$. Since $x \succ y'$ and $x' \succ y'$, thus it is sufficient to consider the condition that (10′) holds. Note that (10′) ⇔

$$p_{12} \leq \frac{x'_2 + x'_3 + x'_4 - (y'_2 + y'_3 + y'_4)}{y_2 + y_3 + y_4 - (y'_2 + y'_3 + y'_4)}. \tag{23}$$

Equation (23) can be rewritten as (21). It can be got by taking $k = y_2 + y_3 + y_4$ in (7).

(III) If $x \succ x'$, $x \parallel y$, $x \parallel y'$, $x' \parallel y$, $x' \parallel y'$ and $y \parallel y'$ (see Fig. 3(b)). It means that one or two of three of inequalities should not hold in the following set of inequalities, respectively:

$$\begin{cases} x_4 \geq y_4, \\ x_3 + x_4 \geq y_3 + y_4, \\ x_2 + x_3 + x_4 \geq y_2 + y_3 + y_4, \end{cases} \qquad \begin{cases} x_4 \geq y'_4, \\ x_3 + x_4 \geq y'_3 + y'_4, \\ x_2 + x_3 + x_4 \geq y'_2 + y'_3 + y'_4, \end{cases}$$

$$\begin{cases} x'_4 \geq y_4, \\ x'_3 + x'_4 \geq y_3 + y_4, \\ x'_2 + x'_3 + x'_4 \geq y_2 + y_3 + y_4, \end{cases} \qquad \begin{cases} x'_4 \geq y'_4, \\ x'_3 + x'_4 \geq y'_3 + y'_4, \\ x'_2 + x'_3 + x'_4 \geq y'_2 + y'_3 + y'_4, \end{cases}$$

$$\begin{cases} y_4 \geq y'_4, \\ y_3 + y_4 \geq y'_3 + y'_4, \\ y_2 + y_3 + y_4 \geq y'_2 + y'_3 + y'_4. \end{cases}$$

Similar to (II), we first discuss the possible options of 0 and 1.

(i) Each column has only one 0 in $(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}$. There are 9 possible options of 0 and 1.

(ii) There exists one and only one column that has two 0 in $(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}$. There are $0+0+15+3+7=20$ possible options of 0 and 1.
(iii) There exist two columns that have two 0 in $(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}$. There are $0+1+0+0+0+0+0+11+9+2=23$ possible options of 0 and 1.
(iv) There exist three columns that have two 0 in $(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}$. There are $0+0+0+26+1+0+7+0+1+30=65$ possible options of 0 and 1.
(v) There exist four columns that have two 0 in $(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}$. There are $4+23+12+0+0=39$ possible options of 0 and 1.
(vi) Each column has two 0 in $(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}$. There are 11 possible options of 0 and 1.

(IV) If $x \parallel x'$, $x \parallel y$, $x \parallel y'$, $x' \parallel y$, $x' \parallel y'$ and $y \parallel y'$ (see Fig. 3(c)). It means that one or two of three of inequalities should not hold in the following set of inequalities, respectively:

$$\begin{cases} x_4' \geq x_4, \\ x_3' + x_4' \geq x_3 + x_4, \\ x_2' + x_3' + x_4' \geq x_2 + x_3 + x_4, \end{cases} \qquad \begin{cases} x_4 \geq y_4, \\ x_3 + x_4 \geq y_3 + y_4, \\ x_2 + x_3 + x_4 \geq y_2 + y_3 + y_4, \end{cases}$$

$$\begin{cases} x_4 \geq y_4', \\ x_3 + x_4 \geq y_3' + y_4', \\ x_2 + x_3 + x_4 \geq y_2' + y_3' + y_4', \end{cases} \qquad \begin{cases} x_4' \geq y_4, \\ x_3' + x_4' \geq y_3 + y_4, \\ x_2' + x_3' + x_4' \geq y_2 + y_3 + y_4, \end{cases}$$

$$\begin{cases} x_4' \geq y_4', \\ x_3' + x_4' \geq y_3' + y_4', \\ x_2' + x_3' + x_4' \geq y_2' + y_3' + y_4', \end{cases} \qquad \begin{cases} y_4 \geq y_4', \\ y_3 + y_4 \geq y_3' + y_4', \\ y_2 + y_3 + y_4 \geq y_2' + y_3' + y_4'. \end{cases}$$

Similar to (II), we first discuss the possible options of 0 and 1.

(i) Each column has only one 0 in $(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}$. There are 108 possible options of 0 and 1.
(ii) There exists one and only one column that has two 0 in $(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}$. There are $30+49+41+120+78+82=400$ possible options of 0 and 1.
(iii) There exist two columns that have two 0 in $(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}$. There are $95+85+24+42+78+30+44+77+33+54+14+23+83+68+24=774$ possible options of 0 and 1.
(iv) There exist three columns that have two 0 in $(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}$. There are $113+42+74+41+34+19+47+18+18+14+39+5+83+19+54+77+92+25+23+6=843$ possible options of 0 and 1.
(v) There exist four columns that have two 0 in $(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}$. There are $91+95+93+36+19+31+46+67+28+72+18+3+17+39+23=678$ possible options of 0 and 1.
(vi) There exist five columns that have two 0 in $(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}$. There are $102+114+57+24+29+74=400$ possible options of 0 and 1.
(vi) Each column has two 0 in $(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}(\cdots)^{\mathrm{T}}$. There are 82 possible options of 0 and 1.

As the argument above, we take $(001)^{\mathrm{T}}(001)^{\mathrm{T}}(110)^{\mathrm{T}}(110)^{\mathrm{T}}(010)^{\mathrm{T}}(110)^{\mathrm{T}}$ as an example. In this case we have $x_4' \geq y_4 > x_4 \geq y_4'$, $x_3' + x_4' \geq y_3 + y_4 > x_3 + x_4 \geq y_3' + y_4'$ and $y_2' + y_3' + y_4' > x_2 + x_3 + x_4 \geq y_2 + y_3 + y_4 > x_2' + x_3' + x_4'$. Now we are to look for the condition that

(8), (9), (10) and (10′) hold. As discussed above, (8)⇔ (17), (9) ⇔ (22), (10) and (10) ⇔ (21), where we take

$$p_{11} \le \min\left\{\frac{x_4 - y_4'}{y_4 - y_4'}, \frac{(x_3+x_4)-(y_3'+y_4')}{(y_3+y_4)-(y_3'+y_4')}, \frac{(x_2+x_3+x_4)-(y_2'+y_3'+y_4')}{(y_2+y_3+y_4)-(y_2'+y_3'+y_4')}\right\}$$

and

$$p_{12} \le \frac{(x_2'+x_3'+x_4')-(y_2'+y_3'+y_4')}{(y_2+y_3+y_4)-(y_2'+y_3'+y_4')}.$$

Equations (17), (22) and (21) can be got by taking $k=1$, y_3+y_4 and 1 in (8), (6) and (7), respectively. In a word, any case can be deduced to the ones that mentioned above.

The general case can be deduced similarly. Let $x_i = (x_{i1}, x_{i2}, \ldots, x_{id})$ and $y_l = (y_{l1}, y_{l2}, \ldots, y_{ld})$ denote the vector of decreasingly ordered Schmidt coefficients of $\mathrm{tr}_A(|\psi_i\rangle\langle\psi_i|)$ and $\mathrm{tr}_A(|\phi_l\rangle\langle\phi_l|)$, respectively. As mentioned above, $\{p_i, |\psi_i\rangle\} \xrightarrow{LOCC} \{q_l, |\phi_l\rangle\}$ if and only if there exists a set of p_{si} such that $x_i \prec \sum_{s=1}^m p_{si} y_s$ for all $1 \le i \le n$. In other words,

$$\sum_{l=j}^{d} x_{il} \ge \sum_{l=j}^{d}\sum_{s=1}^{m} p_{si} y_{sl}, \quad 2 \le j \le d. \tag{24}$$

It is sufficient to obtain (24) from the right hand of (4). So we should consider the possible options according to the partial order of $x_i, y_l (1 \le i \le n, 1 \le l \le m)$. Without loss of the generality, we only consider this case when $x_1, x_2, \ldots, x_{n-1} \prec y_1 \prec x_n \prec y_2, \ldots, y_m$. The other cases can be deduced similarly. Then, it is sufficient to ensure that $x_n \prec \sum_{s=1}^m p_{sn} y_s$ holds. In other words, $\sum_{l=j}^d x_{nl} \ge \sum_{l=j}^d \sum_{s=1}^m p_{sn} y_{sl}$ $(2 \le j \le d)$ have to hold. We take $j = d$ as an example. The other cases can be deduced similarly. Note that $x_{nd} \ge \sum_{s=1}^m p_{sn} y_{sd} \Leftrightarrow$

$$p_{1n} \le \frac{x_{nd} - y_{md}}{y_{1d} - y_{md}} - \sum_{s=2}^{m-1} p_{sn} \frac{y_{sd} - y_{md}}{y_{1d} - y_{md}}. \tag{25}$$

Moreover,

$$\sum_{s=2}^{m-1} \frac{y_{sd} - y_{md}}{y_{1d} - y_{md}} \ge \frac{y_{m-1d} - y_{md}}{y_{1d} - y_{md}} \sum_{s=2}^{m-1} p_{sn} = \frac{y_{m-1d} - y_{md}}{y_{1d} - y_{md}} (1 - p_{1n} - p_{mn}).$$

To make sure that (25) holds, it is sufficient to ensure

$$p_{1n} \le \frac{x_{nd} - y_{md}}{y_{1d} - y_{md}} + p_{mn} \frac{y_{m-1d} - y_{md}}{y_{1d} - y_{md}}. \tag{26}$$

Since $\forall p_{mn} \in [0, 1]$, it is sufficient to make sure

$$p_{1n} \le \frac{x_{nd} - y_{md}}{y_{1d} - y_{md}}. \tag{27}$$

Since $q_1 = \sum_{i=1}^n p_i p_{1i}$, by Lemma 2.10 (27) ⇔

$$\sum_{i=1}^{n-1} p_i + p_n \frac{x_{nd} - y_{md}}{y_{1d} - y_{md}} \ge q_1. \tag{28}$$

Equation (28) can be rewritten as

$$\sum_{i=1}^{n-1} p_i(y_{1d} - y_{md}) + p_n x_{nd} \geq q_1 y_{1d} + \sum_{s=2}^{m} q_s y_{m-1d} + p_n(y_{md} - y_{m-1d}). \tag{29}$$

It is sufficient to make sure

$$\sum_{i=1}^{n-1} p_i(y_{1d} - y_{md}) + p_n x_{nd} \geq q_1 y_{1d} + \sum_{s=2}^{m} q_i y_{m-1d}. \tag{30}$$

Equation (30) can be got by taking $k = y_{1d}$ in the right hand of (4). In a word, in terms of the partial order of x_i, y_l $(1 \leq i \leq n, 1 \leq l \leq m)$ we can always find a proper k such that (24) can be deduced from the right hand of (4).

In summary, we show the necessary and sufficient conditions for entanglement transformation between $d \times d$-dimensional mixed states, which can be regarded as the generalization of Nielsen's theorem. Furthermore, we find that Gour's conjecture is incorrect (see Ref. [7]). We believe that our results will prove fruitful in further developments on mixed state entanglement. Perhaps it also provides a good starting point to better understand the relationship between entanglement and distillation of mixed state.

Acknowledgements We thank all referees for their helpful comments. This work was partially supported by China National Foundation of Natural Science (Grant No. 60134010) and China Postdoctoral Science Foundation(Grant No. 20100471642).

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